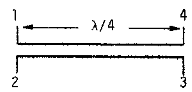


Fig. 1. Conventional directional coupler.

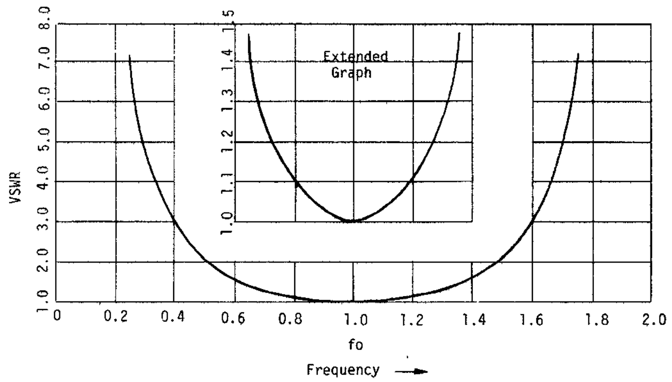


Fig. 2. Input VSWR versus normalized frequency.

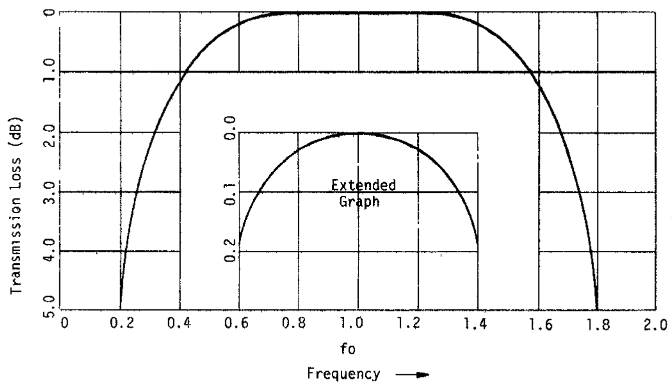


Fig. 3. Transmission loss versus normalized frequency.

or

$$Z_{in1} = \frac{s^2 + (2)^{1/2} s + 1}{s^2 + (2)^{1/2} s} Z_0 \qquad (5)$$

where $k = 1/(2)^{1/2}$ has been used, and $Z_{in1}$ is the input impedance at port 1.

The input VSWR at port 1 can be calculated using (5) as

$$VSWR = \frac{|Z_{in1} + Z_0| + |Z_{in1} - Z_0|}{|Z_{in1} + Z_0| - |Z_{in1} - Z_0|} = 1/\sin^2(\theta). \qquad (6)$$

The transmission loss from port 1 to port 3 can be calculated using (5) as

$$T = 20 \log_{10}\left(1 - \left(\frac{|Z_{in1} - Z_0|}{|Z_{in1} + Z_0|}\right)^2\right)^{1/2} = 20 \log_{10}\left(\frac{2 \sin(\theta)}{1 + \sin^2(\theta)}\right) \text{ (in dB)}. \qquad (7)$$

Equations (6) and (7) are plotted and shown in Figs. 2 and 3, respectively.

Note that Fig. 2 is slightly different from [1, fig. 2]. The differences are due to the different coupling coefficient and the approximate equivalent circuit used in [1].

The other particular case where both port 2 and port 4 are shorted to ground can be analyzed in a similar manner provided that the admittance matrix instead of impedance matrix be used. The input VSWR and the transmission loss can be shown identical to (6) and (7), respectively.

We conclude that dc blocks in microwave frequency can be realized by using $\lambda/4 - 3$-dB directional couplers with both coupled port and transmitted port open-circuited. The exact theoretical responses of input VSWR and transmission loss have been derived.

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# The Synthesis of Quarter-Wave Transformers, Low-Pass and Half-Wave Filters in the Sine Plane

A. I. GRAYZEL

***Abstract*—This letter presents a method for synthesizing quarter-wave Chebyshev transformers, low-pass and half-wave filters. The method uses sine-plane synthesis which greatly simplifies the numerical calculations and allows one to obtain good numerical accuracy using just a desk calculator. Equations for the transmission coefficient are given in a simplified form.**

With the discovery of the $S$-plane equivalent circuit of the transmission line and the development of $S$-plane synthesis techniques [1], [2], it is possible to synthesize quarter-wavelength Chebyshev transformers and low-pass filters with a high degree of accuracy using a desk calculator. This letter presents the equations for the quarter-wave transformer in its simplest form and demonstrates by examples the synthesis procedure for both transformers and low-pass filters.

The equations for the quarter-wavelength transformer have been derived by a number of authors [3]–[5]. The method presented here follows [5] which is believed to be the simplest form. The desired transmission coefficient is given by

$$T = \frac{1}{1 + k^2 T_n^2(x)} \qquad (1)$$

where $T_n$ is the $n$th order Chebyshev polynomial and

$$x = \frac{\cosh \bar{\theta}}{p} \qquad (2a)$$

$$p = \cos \theta_0. \qquad (2b)$$

Then

$$x^2 = \frac{1}{p^2(1 - \lambda^2)} \qquad (2c)$$

where

$$\lambda = \tanh \bar{\theta}. \qquad (2d)$$

$\bar{\theta}$ is a complex angle which on the $j$ axis equals $j\theta$ and $x$ then is equal to $\cos\theta/\cos\theta_0$ where $\theta_0$ and $\pi - \theta_0$ are the cutoff angles of the transformer. The transformer circuit is shown in Fig. 1 and is normalized

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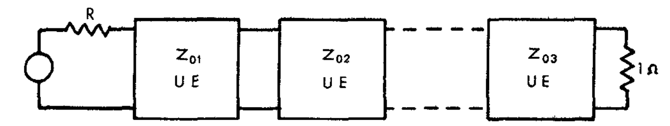


Fig. 1.

to 1 Ω. The transmission coefficient of this circuit at zero frequency is $4R/(1+R)^2$. Equating this to $T(0)$ of (1) yields

$$k^2 = \frac{(R-1)^2}{4RT_n^2(1/p)}. \tag{2e}$$

The zeros of the denominator of (1) are given in [6] as:

$$\text{Im}\,(x_i) = \sin\left(\frac{2i-1}{2n}\pi\right)\sinh\mu, \qquad i = 1,2,\cdots 2n \tag{3a}$$

$$\text{Re}\,(x_i) = \cos\left(\frac{2i-1}{2n}\pi\right)\cosh\mu, \qquad i = 1,2,\cdots 2n \tag{3b}$$

where

$$\left.\begin{matrix}\cosh\mu\\ \sinh\mu\end{matrix}\right\} = \frac{1}{2}\left(\left[\left(\frac{1}{k^2}+1\right)^{1/2}+\frac{1}{k}\right]^{1/n} \pm \left[\left(\frac{1}{k^2}+1\right)^{1/2}+\frac{1}{k}\right]^{-1/n}\right). \tag{3c}$$

It can be seen from (3) that $x_{n+1} = -x_i$ and using (1) the magnitude squared of the reflection coefficient can be written as

$$|p|^2 = 1 - T = \frac{(\alpha k \sum_{k=0}^{n} t_k x^k)^2}{\prod_{i=1}^{n}(x^2 - x_i^2)} \tag{4}$$

where $\alpha$ is a constant and $t_k$ are the coefficients of the Chebyshev polynomial. Substituting for $x^2$ (2c), one obtains the following after some manipulation:

$$\prod_{i=1}^{n}(x^2 - x_i^2) = \frac{(\prod_{i=1}^{n} x_i^2)\prod_{i=1}^{n}(\lambda^2 - \lambda_i^2)}{(1-\lambda^2)^n} \tag{5a}$$

where

$$\lambda_i^2 = 1 - \frac{1}{p^2 x_i^2}. \tag{5b}$$

Equation (4) can then be written as

$$|p|^2 = \frac{A^2\left[[(1-\lambda^2)^{1/2}]^n \sum_{k=0}^{n}\frac{t_k}{p^k}\left(\frac{1}{(1-\lambda^2)^{1/2}}\right)^k\right]^2}{\prod_{i=1}^{n}(\lambda^2 - \lambda_i^2)} \tag{6a}$$

$$= \frac{A^2\left[\sum_{k=0}^{n}\frac{t_k}{p^k}(1-\lambda^2)^{(n-k)/2}\right]^2}{\prod_{i=1}^{n}(\lambda^2 - \lambda_i^2)} \tag{6b}$$

where (2a) has been used to eliminate $x$. It should be noted that when $n$ is even the Chebyshev polynomial is even and hence all the $t_k$ for $k$ odd are zero. Similarly when $n$ is odd all the $t_k$ for $k$ even are zero. It therefore follows that for all terms in the numerator of (6b) whose coefficients are nonzero $(n-k)/2$ is an integer. The numerator of (6b) is therefore a perfect square. $p(\lambda)$ is formed by taking the square root of the numerator and the left half-plane (LHP) zeros of the denominator. Thus

$$p(\lambda) = \frac{A\sum_{k=0}^{n}\frac{t_k}{p^k}(1-\lambda^2)^{(n-k)/2}}{\prod_{i=1}^{n}(\lambda - \bar{\lambda}_i)} = \frac{A\sum_{j=0}^{m} a_j\lambda^{2j}}{\prod_{i=1}^{n}(\lambda - \bar{\lambda}_i)},$$

$$m = \begin{cases}\dfrac{n}{2} & n \text{ even}\\[2ex] \dfrac{n-1}{2} & n \text{ odd.}\end{cases} \tag{7}$$

The $\lambda_i$ are given by (5b). We must select those zeros in the LHP. It can be shown that the zeros in the LHP can be generated from the $x_i$ in the upper right half-plane by the relationship

$$\bar{\lambda}_i = -\left(1 - \frac{1}{p^2 x_i^2}\right)^{1/2}, \qquad i = 1,2,\cdots,l$$

$$\bar{\lambda}_{n-1+i} = \bar{\lambda}_i^* \qquad l = \begin{cases}\dfrac{n+1}{2} & n \text{ odd}\\[2ex] n/2 & n \text{ even.}\end{cases} \tag{8}$$

Let us rewrite (7) as

$$p(\lambda) = \frac{A\sum_{j=0}^{m} a_j\lambda^{2j}}{\sum_{i=0}^{n} b_i\lambda^i}. \tag{9}$$

We must now determine the coefficient $A$. We will normalize the transformer to a load of 1 Ω as shown in Fig. 1. The reflection coefficient at zero frequency is then $(R-1)/(R+1)$, which must therefore be the value of $p(0)$ given by (9). Equating we find that

$$A = \frac{b_0(R-1)}{a_0(R+1)}. \tag{10}$$

Finally the input impedance $Z(\lambda)$ of the transformer of Fig. 1 is given by

$$Z(\lambda) = R\frac{1-p}{1+p} = R\frac{\sum_{i=0}^{n} b_i\lambda^i - A\sum_{j=0}^{m} a_j\lambda^{2j}}{\sum_{i=0}^{n} b_i\lambda^i + A\sum_{j=0}^{m} a_j\lambda^{2j}}. \tag{11}$$

As discussed in [2] $z_{22}$ is the ratio of the even to odd parts of the denominator of the input impedance given by (11). Having determined $z_{22}$ we synthesize the network in the $S$ plane as described in [2].

We will now solve the same example given in [4]. We will match a resistance of 0.44 to a 1-Ω load with $p = 0.464$. We find from (2e) that $k^2 = 1.5777 \times 10^{-4}$ and from (3) that $\cosh\mu = 2.80233$, $\sinh\mu = 2.61783$, $x_1 = 2.42688 + j1.30891$, and $x_2 = j2.61783$. Substitution into (8) yields $\bar{\lambda}_1 = -0.86667 - j.29450$ and $\bar{\lambda}_2 = -1.29529$. Multiplying out the three factors $(\lambda - \bar{\lambda}_1)(\lambda - \bar{\lambda}_1^*)(\lambda - \bar{\lambda}_2)$ we find that $b_3 = 1$, $b_2 = 3.02863$, $b_1 = 3.08303$, and $b_0 = 1.08526$. We must now evaluate the numerator polynomial of (7). The coefficients of $T_3$ are $t_1 = -3$ and $t_3 = 4$, which yield for $p = 0.464$, $a_0 = 33.57559$, and $a_1 = 6.455$. Evaluating (10) we find $A = -0.01257$. The denominator polynomial of $Z(\lambda)$ (11) is then

$$\lambda^3 + 2.94736\lambda^2 + 3.08303\lambda + 0.66321$$

and

$$z_{22} = \frac{2.94736\lambda^2 + 0.66321}{\lambda^3 + 3.08303\lambda}.$$

To synthesize the network we will form $z_{22}''(S) = z_{22}/C$ as described in [2] and synthesize the network in the form shown in Fig. 2.

$$D_n\lambda^n + D_{n-1}\lambda^{n-1}\cdots + D_0$$

then replacing $\lambda$ by $1/\lambda$ one obtains the denominator polynomial for the low-pass filter $D_0\lambda^n + D_1\lambda^{n-1}\cdots + D_n$. One then forms $z_{22}$ as the even over the odd part of the polynomial, and synthesizes the net-

$$z_{22}'' = \frac{2.94736\dfrac{S^2}{S^2+1} + 0.66321}{S\left(\dfrac{S^2}{S^2+1} + 3.08303\right)} = \frac{3.61057S^2 + 0.66321}{4.08303S^3 + 3.08303S}$$

$$\begin{array}{l} 3.61057S^2 + 0.66321 \;\big)\; 4.08303S^3 + 3.08303S \;\big(\; 1.13085S \\ \qquad\qquad\qquad\qquad\;\; \underline{4.08303S^3 + 0.74994S} \\ \qquad\qquad\qquad\qquad\qquad\quad 2.33309S \;\big)\; 3.61057S^2 + 0.66321 \;\big(\; 1.54755S \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{3.61057S^2} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0.66321 \;\big)\; 2.33309S \;\big(\; 3.51788S. \end{array}$$

The resultant network is that of Fig. 3 with $C_1 = 3.51788$, $L_2 = 1.54755$, and $C_2 = 1.13085$. $L_1$ has not been determined but is not necessary. We divide the elements as shown in Fig. 4 to be compatible with Fig. 2 and find that $L_1$ must equal 0.49747. The resulting transmission-line network is then the network of Fig. 5.

The transmission coefficient of a low-pass or half-wave filter made up of cascaded unit elements is also given by (1) where $x = \sinh \bar{\theta}/jp$ and $p = \sin \theta_c$. $\bar{\theta}$ is a complex angle which on the $j$ axis equals $j\theta$ and $x$ then equals $\sin \theta/\sin \theta_c$. $\theta_c$ is the cutoff angle of the low-pass filter and $2\theta_c/\pi$ is the percentage bandwidth of the half-wave filter. For the same value of $k$ and $\sin \theta_c = \cos \theta_0$ ($\theta_c = \pi 2 - \theta_0$) the transmission coefficients are identical except for a translation of 90°. Cristal has shown [7] that the input impedance of one can be derived from the other under this condition by replacing $\lambda$ by $1/\lambda$. If one has found the denominator polynomial of (11)

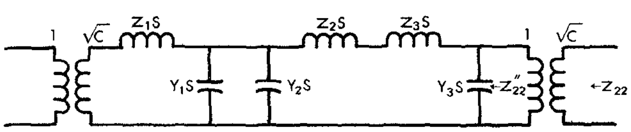


Fig. 2.

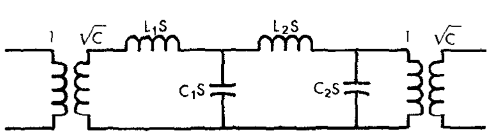


Fig. 3.

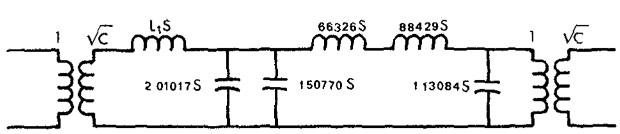


Fig. 4.

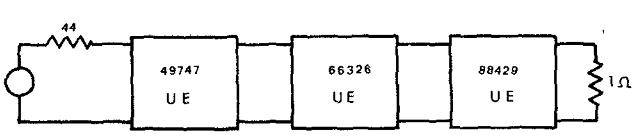


Fig. 5.

work in the $S$ plane exactly as was done in the previous example. Hence if one wants to synthesize a three-section Chebyshev low-pass filter with $k^2 = 1.5777 \times 10^{-4}$ and $\sin \theta_c = 0.464$ then one obtains directly from the coefficients of the denominator polynomial of the previous example that

$$z_{22} = \frac{3.08303\lambda^2 + 1}{0.66321\lambda^3 + 2.94736\lambda}.$$

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# Single Hybrid Tee Frequency Discriminator

J. NIGRIN, N. A. MANSOUR, AND W. A. G. VOSS

***Abstract*—A novel single hybrid tee frequency discriminator is investigated. It consists of ordinary microwave components and its tuning is achieved by means of a movable short circuit. The discriminator properties are comparable to those of an ordinary phase dis-**

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